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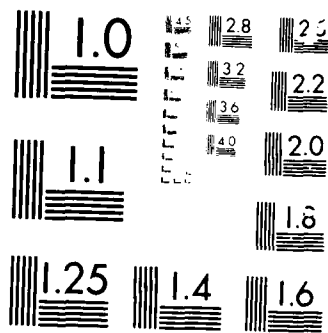
MATCHING EXTENSION IN BIPARTITE GRAPHS 1 INTRODUCTION 1/1
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MATCHING EXTENSION IN BIPARTITE GRAPHS

by

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1. Introduction and Terminology

All graphs in this paper will be finite and connected and will have no loops or parallel lines.

Let n and p be positive integers with $n \leq (p-2)/2$ and let G be a graph with p points having a perfect matching. Graph G is said to be **n -extendable** if every matching of size n in G extends to a perfect matching. In this paper, we will be concerned primarily with studying n -extendability in bipartite graphs.

Let us begin, however, with a few historical remarks. The concept of n -extendability seems to have its earliest roots in a paper of Hetyei (1964) who studied the concept for bipartite graphs. In this early paper, Hetyei obtained three different characterizations of 1-extendable bipartite graphs. Lovász and the present author (1977) gave a fourth such characterization which they referred to as an "ear structure theorem". Unknown to them, however, Hartfiel (1970) had already formulated an equivalent theorem, but couched in terms of matrices. A year later, Brualdi and Perfect (1971) published a paper in which they gave the first characterization of n -extendable bipartite graphs, but they too couched their results in terms of matrices ("extending partial diagonals") and set systems ("extending partial systems of distinct representatives (PSDR's)").

Approaching from yet another direction, Berge (1978a, 1978b, 1978c 1981) introduced the concept of a **regularizable** graph. A graph G is said to be **regularizable** if by adding new parallel lines to certain of those already existing in G , one can obtain a regular multigraph. Berge was led to this idea from considerations in what is sometimes called "fractional transversal theory". In (1981) Berge proved a result equivalent to showing that in the bipartite graph case, the property of being regularizable is the same as being 1-extendable.

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Motivated by the results of Kotzig, Gallai and Edmonds, Lovász (1972) extended and refined the canonical decompositions already extant while analyzing further the structure of graphs which are *elementary*, thus extending the earlier work of Heteyi and Kotzig. A graph G is called **elementary** if the set of its lines which lie in at least one perfect matching form a connected subgraph of G . (In the bipartite case, we shall see below that the properties of being elementary and of being 1-extendable are equivalent.)

In this same paper, Lovász also introduced the concept of a *bicritical* graph. A graph G is said to be **bicritical** if $G - u - v$ has a perfect matching for every pair of distinct points u and v in $V(G)$. In the last ten years or so, the earlier work on decompositions of graphs in terms of their matchings has evolved further (see Lovász and Plummer (1986)) and today much attention continues to be focused upon the structure of bicritical graphs which are, in addition, 3-connected. Such graphs have been christened **bricks**. (See, for example, the paper by Edmonds, Lovász and Pulleyblank (1982) and that of Lovász (1986).)

But what is the connection between n -extendability and bicriticality? In 1980, the author published a paper on general n -extendable graphs. One of the results presented in that paper states that every 2-extendable graph is either bipartite or is a brick. (The reader should convince himself immediately that these two classes of graphs are disjoint.) Motivated by this result, the author has continued to study properties of n -extendable graphs (see (1985, 1986a, 1986b and 1986c)).

In the first section of this paper we gather together the various characterizations of 1-extendable bipartite graphs mentioned above and then give the natural generalizations to n -extendability with a unified proof of the equivalencies. 21

In (1986b), the present author presents some results on the connectivity of general n -extendable graphs. In the second section of the

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present paper, we prove a result about connectivity which is peculiar to the bipartite case.

All graph terminology not defined in this paper may be found in Bondy and Murty (1976) and Lovász and Plummer (1986).

2. Characterizations of 1-extendable Bipartite Graphs

In addition to the theorem of the author (1980) mentioned in the Introduction, there are two other results proved in that paper which we shall use here and hence we begin by stating them without proof.

1980A. THEOREM. *If $n \geq 2$ and G is n -extendable, then G is also $(n - 1)$ -extendable.* ■

1980B. THEOREM. *If G is n -extendable, then G is $(n + 1)$ -connected.* ■

We begin with the various characterizations known for 1-extendable graphs. But first we must explain what we shall mean by an *ear construction*.

An Ear Construction. Let e be any line. Join its endpoints by a path ϵ_1 of odd length (the so-called "first ear"). Inductively one may construct a sequence of bipartite graphs as follows. If $G = e + \epsilon_1 + \epsilon_2 + \dots + \epsilon_{r-1}$ has already been constructed, add the r th ear ϵ_r by joining any two points of G which lie in different sets of the bipartition by a path (ϵ_r) of odd length.

2.1. THEOREM. *Let G be a bipartite graph with bipartition (U, W) . Then the following are equivalent:*

- (i) G is elementary;
- (ii) $G = K_2$ or $|V(G)| \geq 4$ and for all points $u \in U$ and $w \in W$, $G - u - w$ has a perfect matching;
- (iii) $|U| = |W|$ and for all non-empty proper subsets X of U , $|\Gamma(X)| \geq |X| + 1$;
- (iv) G is connected and 1-extendable;
- (v) G may be obtained by an ear construction;
- (vi) G is regularizable.

PROOF. (Outline). The equivalence of the first four statements are essentially due to Heteyi (1964) and are also proved in Lovász and Plummer (1977; see also 1986, Theorem 4.1.1) as is the equivalence of (iv) and (v). The equivalence of (iv) and (v), however, were demonstrated earlier

by Hartfiel (1970) expressed in the different terminology of indecomposable and nearly decomposable matrices.

The equivalence of (iii) and (vi) is due first to Berge (1981), but we shall present a proof essentially identical to that of Lovász and Plummer (1986, Theorem 6.2.9).

Suppose first that G is regularizable. Since a graph is regularizable if and only if each of its connected components is, we may assume that G is connected. Let us assume, however, that there is a non-empty proper subset X of U for which $|\Gamma(X)| \leq |X|$. Suppose, then, that we choose a *minimal* subset A of U with this property. Let G_1 result from regularizing G and suppose $\deg_{G_1} v = d$. Now there exist $d|A|$ lines out of A and hence into $\Gamma(A)$ and thus there can be no lines from $U - A$ to $\Gamma(A)$. But then by connectivity, $U - A = \emptyset$ and hence $W - \Gamma(A) = \emptyset$. So $U = A$ and $W = \Gamma(A)$. But by the minimality of A , we have that all non-empty proper subsets Y of A ($= U$), $|\Gamma(Y)| > |Y|$. Thus (iii) is satisfied.

Now conversely, suppose that G is a 1-extendable bipartite graph. Let the lines of G be e_1, \dots, e_q . For each of these lines e_i let F_i be a perfect matching containing e_i . Form a multigraph G^* by taking the *disjoint* union of F_1, F_2, \dots, F_q . Then G^* is a q -regular multigraph and hence G is regularizable. ■

Now we present the generalization of the preceding theorem to the case of n -extendable bipartite graphs. The equivalence of (i) and (ii) in this theorem was first demonstrated by Brualdi and Perfect (1971), but was done in the language of set systems (completing partial systems of distinct representatives). We present a proof by induction in which we use the Theorems 1980A and 1980B stated above.

2.2. THEOREM. *Let G be a connected bipartite graph with bipartition (U, W) , with $p = |V(G)|$ and suppose n is a positive integer such that $n \leq (p - 2)/2$. Then the following are equivalent:*

- (i) G is n -extendable;
- (ii) $|U| = |W|$ and for all non-empty subsets X of U , $|\Gamma(X)| \geq |X| + n$;
- (iii) For all $u_1, \dots, u_n \in U$ and $w_1, \dots, w_n \in W$, $G' = G - u_1 - \dots - u_n - w_1 - \dots - w_n$ has a perfect matching.

PROOF. (i) \Rightarrow (ii). The proof is by induction on n . If $n = 1$, we are finished by Theorem 2.1. So suppose $n \geq 2$ and that the implication is true for $1, \dots, n - 1$. Suppose that X is a non-empty proper subset of U with $|X| \leq n$, but $|\Gamma(X)| \leq |X| + (n - 1)$. Since G is $(n - 1)$ -extendable

by Theorem 1980A, we know that $|\Gamma(X)| \geq |X| + (n-1)$ by the induction hypothesis, so, in fact, we must have $|\Gamma(X)| = |X| + (n-1)$.

Now $|\Gamma(X)| = |X| + n - 1 \geq n$, since $X \neq \emptyset$. Moreover, $|U - X| = |U| - |X| \geq n$ by the hypothesis on the size of set X . Finally, graph G is n -connected by Theorem 1980B.

First let us suppose that $W - \Gamma(X) \neq \emptyset$. Choose any point $x \in X$ and a point $y \in W - \Gamma(X)$. By Whitney's well-known theorem on connectivity, there must exist at least n point-disjoint paths in G which join points x and y . But these paths must then contain a matching of size at least n between $\Gamma(X)$ and $U - X$. Let this matching be denoted by $e_1, \dots, e_n, e_{n+1}, \dots$. Since G is n -extendable, $\{e_1, \dots, e_n\}$ must extend to a perfect matching of G which must match $|X| + n$ points into $\Gamma(X)$. Hence $|\Gamma(X)| \geq |X| + n$, a contradiction.

So we may assume that $W - \Gamma(X) = \emptyset$. In this case, add a new "phantom point" to G , call it w^* and join it to all points in $U - X$. It is an easy matter to see that the new resulting (bipartite) graph G^* is also n -connected. Now select a point $x \in X$ together with point w^* and, applying Whitney's Theorem, proceed as before. This completes the proof of (i) \Rightarrow (ii).

(ii) \Rightarrow (iii). Suppose $|U| = |W|$ and let G' be as defined as in the statement of the theorem. Let $U' = U - u_1 - \dots - u_n$ and $W' = W - w_1 - \dots - w_n$. Choose any $X' \subseteq U'$, where $X' \neq \emptyset$. Then $|X'| \leq |U'| = |U| - n$, and so $|\Gamma_G(X')| \geq |X'| + n$ and hence $|\Gamma_{G'}(X')| \geq |G'|$. Now $|U'| = |W'|$ and hence by Philip Hall's classical theorem on bipartite matching applied to graph G' , G' must have a perfect matching. This proves (ii) \Rightarrow (iii).

(iii) \Rightarrow (i) is trivial. ■

It may be shown (see Plummer (1986b)) that if a line e is deleted from any (not necessarily bipartite) n -extendable graph G , then $G - e$ is $(n-1)$ -extendable. But what happens if one *adds* a new line to an n -extendable graph? Here the outcome can be dramatically different, depending upon whether or not graph $G + e$ is bipartite. It is possible, for example, to add a line to an n -extendable graph and have the resulting graph fail to be even 1-extendable! A simple example of this behavior is furnished by adding a line joining two points of the same set of the bipartition of the complete bipartite graph $K_{n+1, n+1}$. On the other hand, we have the following corollary to Theorem 2.2. (Here \bar{G} denotes the *complement* of G .)

2.3. COROLLARY. *Suppose n is a positive integer and G is an n -*

extendable bipartite graph. Let e be a line in \overline{G} such that $G + e$ is still bipartite. Then $G + e$ is also n -extendable.

PROOF. The proof is immediate from either part (ii) or part (iii) of Theorem 2.2. ■

We then have the following result concerning deletion of lines from n -extendable bipartite graphs.

2.4. COROLLARY. *Suppose $n \geq 2$ and let G be an n -extendable bipartite graph with point bipartition $U \cup W$. Then if $u \in U$ and $w \in W$, the graph $G - u - w$ is $(n - 1)$ -extendable.*

PROOF. Graph G is 1-extendable by Theorem 1980A and hence by part (ii) of Theorem 2.1, $G - u - w$ has a perfect matching. Thus it contains sets of $n - 1$ independent lines.

Let $e_1 = u_1w_1, \dots, e_{n-1} = a_{n-1}b_{n-1}$ be a set of $n - 1$ independent lines in $G' = G - u - w$.

First suppose line $e = uw \in E(G)$. Then $\{e_1, \dots, e_{n-1}, e\}$ is a set of n independent lines in G . So let F_1 be a perfect matching in G containing these n lines. Then $F_1 - uw$ is a perfect matching for $G - u - w$ containing lines e_1, \dots, e_{n-1} .

So suppose that $e = uw \notin E(G)$. Then by Corollary 2.3, graph $G + e$ is n -extendable. Let F_2 be a perfect matching for $G + e$ which contains lines e_1, \dots, e_{n-1} and e . Then $F_2 - e$ is a perfect matching for G which does not touch points u or w and hence $F_2 - e$ is a perfect matching for $G' = G - u - w$ and it contains lines e_1, \dots, e_{n-1} . Hence G' is $(n - 1)$ -extendable as claimed. ■

The next corollary then follows immediately from Corollary 2.4 and Theorem 1980B.

2.5. COROLLARY. *Suppose $n \geq 2$ and let G be an n -extendable bipartite graph with point bipartition $U \cup W$. Then if $u \in U$ and $w \in W$, the graph $G - u - w$ is n -connected.* ■

And finally the next result follows immediately by repeated applications of Corollary 2.4 and Theorem 1980B.

2.6. COROLLARY. *Suppose $n \geq 2$ and $0 < r < n$. Suppose that G is an n -extendable bipartite graph with point bipartition (U, W) and that $u_1, \dots, u_r \in U$ and $w_1, \dots, w_r \in W$. Then graph $G - u_1 - \dots - u_r - w_1 - \dots - w_r$ is $(n - r)$ -extendable and hence $(n - r + 1)$ -connected. ■*

3. A Connectivity Result

We present a result on line cutsets of n -extendable graphs which is peculiar to the bipartite case. Let us define a **fan** to be any subgraph of a graph which is isomorphic to the complete bipartite graph $K_{1,r}$ for some $r \geq 1$.

3.1. THEOREM. *Let n be a positive integer and let G be an n -extendable bipartite graph with bipartition $V(G) = U \cup W$. Let S be a line cutset of G which is not a fan. Then:*

- (a) *if $n = 1$, $|S| \geq 2$, while*
- (b) *if $n \geq 2$, then $|S| \geq n + 2$.*

PROOF. By Theorem 1980B, we have $n + 1 \leq \kappa(G) \leq \lambda(G) \leq |S|$ and hence if $n = 1$, $|S| \geq 2$ as claimed.

Now suppose $n \geq 2$. Suppose, contrary to the hypothesis, that $|S| \leq n + 1$. Again by Theorem 1980B, we must have $|S| = n + 1$. But since S is not a fan, it is easy to see that S must be a matching. Let $S = \{e_1, \dots, e_{n+1}\}$. Since S is a minimum cutset, $G - S$ consists of exactly two components G_1 and G_2 .

Now suppose the bipartition of G is $U \cup W$. For $i = 1, 2$, let $U_i = V(G_i) \cap U$, $W_i = V(G_i) \cap W$, $U'_i = (V(G_i) \cap U) - U_i$ and $W'_i = (V(G_i) \cap W) - W_i$. Interchanging the names of U and W if necessary, we may assume $|U_1| \geq |W_1|$. Let $e_i = u_i w_i$ for $i = 1, \dots, n + 1$ and renumbering the e_i 's if necessary, we may suppose that $u_1, \dots, u_j \in U_1$, $u_{j+1}, \dots, u_{n+1} \in U_2$, $w_1, \dots, w_j \in W_2$ and $w_{j+1}, \dots, w_{n+1} \in W_1$.

Now suppose that $W'_1 = \emptyset$. Then since $\deg u_1 \geq n + 1$ and since $|U_1 \cup W_1| = n + 1$, we must have u_1 adjacent to all points of $(U_1 \cup W_1) - u_1$ and hence to all points of $U_1 - u_1$. But then $|U_1| = 1$ (i.e., $U_1 = \{u_1\}$) and $|W_1| \leq 1$. Thus $n + 1 \leq 2$ or $n \leq 1$, a contradiction.

Thus $W'_1 \neq \emptyset$. But then $G_1 \neq \emptyset$ and $U_1 \cup W_1$ is a point cutset of G of size $n + 1$ and hence a minimum cutset of G . It then follows by Theorem 2.2 of Plummer (1986b) that S must be an independent set.

Claim 1. *There is a complete matching of U_1 into W'_1 .*

Suppose there is no such matching. Then by Philip Hall's classical theorem on bipartite matching, there is a set $U_0 \subseteq U_1$ with $|\Gamma_{G_1}(U_0)| < |U_0|$. Suppose now that $V(G_1) - \Gamma_{G_1}(U_0) \neq \emptyset$. Then $\Gamma_{G_1}(U_0) \cup (U_1 - U_0) \cup W_1$ is a cutset of G with $|\Gamma_{G_1}(U_0) \cup (U_1 - U_0) \cup W_1| = |\Gamma_{G_1}(U_0)| + |U_1 - U_0| + |W_1| < |U_0| + |U_1 - U_0| + |W_1| = |U_1| + |W_1| = |U_1 \cup W_1|$, contradicting the fact that $U_1 \cup W_1$ is a *minimum* cutset of G .

Thus we may suppose $V(G_1) - \Gamma_{G_1}(U_0) = \emptyset$; that is, $\Gamma_{G_1}(U_0) = V(G_1)$. So in particular, $U'_1 = \emptyset$.

Suppose now that $W_1 \neq \emptyset$. Then we have $\deg_G w_1 \geq n+1$ and since S is independent, $|U'_1| \geq n \geq 2$. In particular, $U'_1 \neq \emptyset$, a contradiction.

So we may suppose that $W_1 = \emptyset$. Now extend $\{e_1, \dots, e_n\}$ to a perfect matching F_1 of G . If $e_{n+1} \in F_1$, we would have $|W'_1| = |U'_1|$, a contradiction since $W'_1 \neq \emptyset$ while $U'_1 = \emptyset$. Thus $e_{n+1} \notin F_1$ and F_1 must match u_{n+1} to a point $w \in W'_1$ and again, since $U'_1 = \emptyset$, it follows that $|W'_1| = 1$. But then $\deg u_1 \leq 2 < \kappa(G)$, a contradiction.

So for all $U_0 \subseteq U_1$, we have $|\Gamma_{G_1}(U_0)| \geq |U_0|$. Thus by Hall's Theorem, there is a matching of U_1 into W'_1 and Claim 1 is proved.

Claim 2. If $W_1 \neq \emptyset$, there is a complete matching of W_1 into U'_1 .

Suppose $w_k \in W_1$. Then $\deg w_k \geq n+1 \geq 3$, and since S is independent, $U'_1 \neq \emptyset$.

Now suppose that no complete matching of W_1 into U'_1 exists. Then by Hall's Theorem, there must exist a set $W_0 \subseteq W_1$ with $|\Gamma_{G_1}(W_0)| < |W_0|$. Now since $W'_1 \neq \emptyset$, $\Gamma_{G_1}(W_0) \cup (W - W_0) \cup U_1$ is a point cutset of G and $|\Gamma_{G_1}(W_0) \cup (W_1 - W_0) \cup U_1| = |\Gamma_{G_1}(W_0)| + |W_1| - |W_0| + |U_1| < |W_0| + |W_1| - |W_0| + |U_1| = |U_1| + |W_1| = |U_1 \cup W_1|$, contradicting the minimality of cutset $U_1 \cup W_1$.

Thus for all $W_0 \subseteq W_1$, we have $|\Gamma_{G_1}(W_0)| \geq |W_0|$ and once again by Hall's Theorem, there is a complete matching of W_1 into U'_1 as claimed. This completes the proof of Claim 2.

Now suppose $W_1 = \emptyset$. Extend $\{e_1, \dots, e_n\}$ to a perfect matching F_2 of G .

First suppose $e_{n+1} \in F_2$. Then

$$|U'_1| = |W'_1|. \quad (3.1)$$

Now by Claim 1, there is a matching of U_1 into W'_1 ; call it f_1, \dots, f_{n+1} . Extend this matching to a perfect matching F_3 of G . But then if $e_{n+1} \notin F_3$, we have $|W'_1| - (n+1) = |U'_1|$ and by equation (3.1) it follows that $n \leq -1$ which is a contradiction. On the other hand, if $e_{n+1} \in F_3$, we have $|W'_1| - n = |U'_1|$ and again by equation (3.1) we have $n = 0$ which is also a contradiction.

Thus we may suppose $e_{n+1} \notin F_2$ and

$$|W'_1| - 1 = |U'_1|. \quad (3.2)$$

Again by Claim 1 there is a matching of U_1 into W'_1 , f_1, \dots, f_{n+1} . Extend f_1, \dots, f_n to a perfect matching F_4 of G . If $e_{n+1} \notin F_4$, we have $|W'_1| - (n+1) = |U'_1|$ and by equation (3.2) it follows that $n = 0$, another

contradiction. On the other hand, if $e_{n+1} \in F_4$, we have $|W'_1| - n = |U'_1|$ and by equation (3.2), $n = 1$, again a contradiction.

So we may suppose $W_1 \neq \emptyset$. Thus $j \leq n$ and also if $w \in W_1$, then $\deg w \geq n + 1 \geq 3$ and hence $|U'_1| \geq 2$.

Now by Claim 2, there is a matching f_{j+1}, \dots, f_{n+1} of W_1 into U'_1 and by Claim 1, there is a matching f_1, \dots, f_j of U_1 into W'_1 . Extend $\{f_1, \dots, f_j, f_{j+1}, \dots, f_n\}$ to a perfect matching F_5 of G . Relabeling if necessary, we may assume that f_i is adjacent to line e_i for all $i = 1, \dots, n + 1$.

First suppose that F_5 matches w_{n+1} into U'_1 . Then $|W'_1| - j = |U'_1| - (n + 1 - j)$ or

$$|W'_1| = |W'_1| = |U'_1| - n - 1 + 2j. \quad (3.3)$$

Now extend $\{e_1, \dots, e_n\}$ to a perfect matching F_6 of G .

Suppose first that $e_{n+1} \in F_6$ also. Then we have

$$|W'_1| = |U'_1| \quad (3.4)$$

and by equation (3.3), $2j = n + 1$ or $j = (n + 1)/2$. So we have that $|U_1| = |W_1|$, n is odd and $n \geq 3$, and also $j \geq 2$.

Now extend $\{f_1, \dots, f_j, e_{j+1}, \dots, e_n\}$ to a perfect matching F_7 of G . If F_7 matches w_{n+1} into U'_1 , we have $|W'_1| - j = |U'_1| - 1$ and by equation (3.4) we have $j = 1$, a contradiction. On the other hand, if F_7 uses e_{n+1} , we have $|W'_1| - j = |U'_1|$ and by equation (3.4) we get that $j = 0$ which is again a contradiction.

Thus we may suppose that $e_{n+1} \notin F_6$ and hence F_6 matches w_{n+1} into U'_1 . So we now have

$$|W'_1| = |U'_1| - 1. \quad (3.5)$$

Comparing this with equation (3.3) we find that $2j = n$ and thus $j = n/2$ and n is even.

Let us now extend the matching $\{f_1, \dots, f_j, e_{j+1}, \dots, e_n\}$ to a perfect matching F_8 of G . First suppose that F_8 does not contain e_{n+1} and hence matches point w_{n+1} into U'_1 . Then $|W'_1| - j = |U'_1| - 1$ or $|W'_1| = |U'_1| + j - 1$. But comparing this with equation (3.5) we have $j = 0$, a contradiction.

On the other hand, suppose F_8 contains line e_{n+1} . Then we have $|W'_1| - j = |U'_1|$ and again by equation (3.5) we have $j = -1$, another contradiction.

So we now suppose that perfect matching F_5 contains line e_{n+1} . Then $|W'_1| - j = |U'_1| - (n - j)$ or

$$|W'_1| = |U'_1| - n + 2j. \quad (3.6)$$

Now extend $\{e_1, \dots, e_n\}$ to a perfect matching F_9 of G .

Suppose first that $e_{n+1} \in F_9$. Then we have $|W'_1| = |U'_1|$ and by equation (3.6), $n = 2j$ or $j = n/2$. But this contradicts the fact that $|U_1| \geq |W_1|$.

So finally suppose that $e_{n+1} \notin F_9$. So F_9 matches w_{n+1} into $|U'_1|$. Thus $|W'_1| = |U'_1| - 1$ and by equation (3.6), we have $-1 = -n + 2j$ or $j = (n - 1)/2$, again contradicting the fact that $|U_1| \geq |W_1|$. ■

Let us recall at this point the concept of *cyclic connectivity* in a graph. A graph G is said to have **cyclic connectivity** $= n$ if among all line cutsets of G which leave two components each containing a cycle, n is the cardinality of a smallest one. We shall denote the cyclic connectivity of G by $c\lambda(G)$.

The above theorem yields an immediate corollary involving the cyclic connectivity of an n -extendable bipartite graph.

3.2. COROLLARY. *Let G be a bipartite n -extendable graph. Then*

- (a) *if $n = 1$, $c\lambda(G) \geq 2$, while*
- (b) *if $n \geq 2$, then $c\lambda(G) \geq n + 2$.*

PROOF. We know that $2 \leq n + 1 \leq \kappa(G) \leq \lambda(G) \leq c\lambda(G)$. Suppose that S is a line cutset the deletion of which leaves two components each containing a cycle. Then S cannot be a fan and the result follows by the preceding theorem. ■

REMARK 1. Let us note that the demand that graph G be bipartite is necessary for the bound of Theorem 3.1. In Figure 3.1 we show an infinite family of graphs $G(n)$ such that for each $n \geq 1$, graph $G(n)$ is n -extendable and G has an (independent) line cutset S of cardinality exactly $n + 1$. (Note that in this figure the large "+" sign means the join operation in which all points in the left-hand K_{3n} are joined by lines to all points of U and similarly, all points of W are joined to all points of the right-hand K_{3n} .)

Verification that $G(n)$ is n -extendable is quite easy and we leave this to the reader.

FIGURE 3.1. The family $\{G(n)\}$

4. Some Concluding Remarks

Several interesting questions remained unanswered about n -extendable bipartite graphs. For example, we do not claim that the bound of Corollary 3.2 is sharp. We conjecture that in fact, if G is a bipartite n -extendable graph (with $n \geq 3$) that the cyclic connectivity may be as high as $2n - 1$. We have proved this for $n = 3$ and 4 using the methods of the present paper. However, these methods seem to fail for $n \geq 5$ and some new ideas appear to be necessary.

The upper bound of $2n - 1$ is best possible for the cyclic connectivity of a bipartite n -extendable graph (where n is at least 3). This is shown by the infinite family of graphs $\{H(n)\}_{n=3}^{\infty}$ shown in Figure 3.2. In graph $H(n)$ we have $|U_1| = |W_2| = n$, $|W_1| = |U_2| = n - 1$ and $|U'_1| = |W'_1| = |U'_2| = |W'_2| = n$.

The bound of 4 for the cyclic connectivity is sharp when $n = 2$ as well and an extremal graph is shown in Figure 3.3.

Finally, we ask if there is some kind of "ear structure" for 2-extendable bipartite graphs (or more generally, for n -extendable bipartite graphs) in some way parallel to that for 1-extendable bigraphs.

FIGURE 3.2. The family $\{H(n)\}_{n=3}^{\infty}$

FIGURE 3.3.

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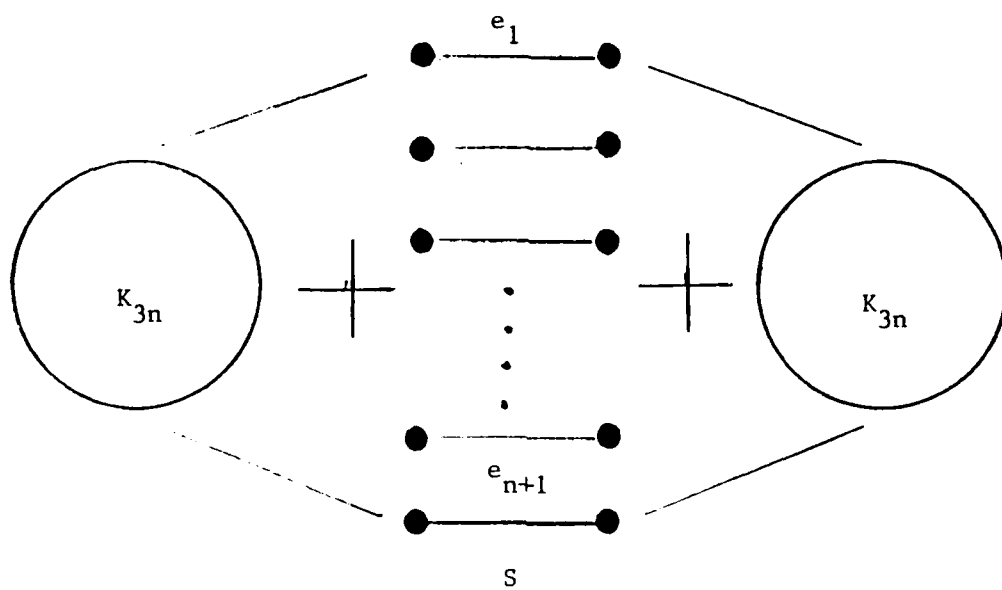
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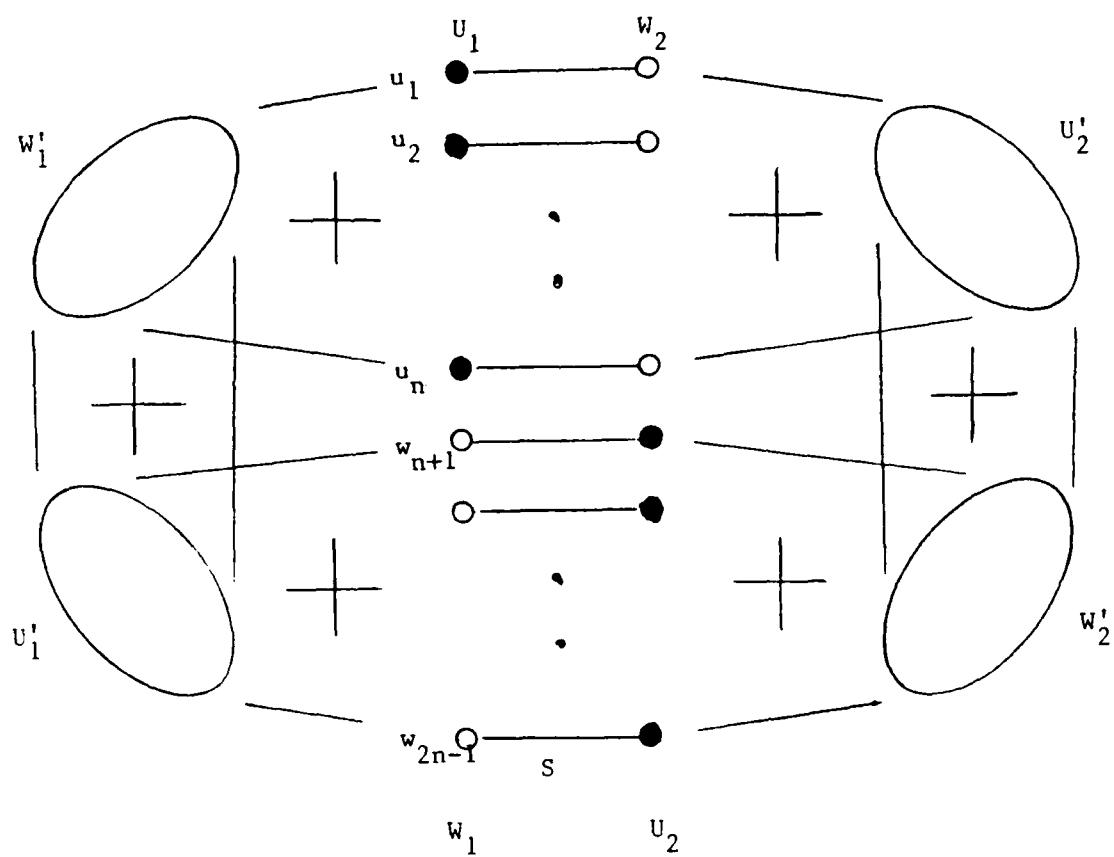
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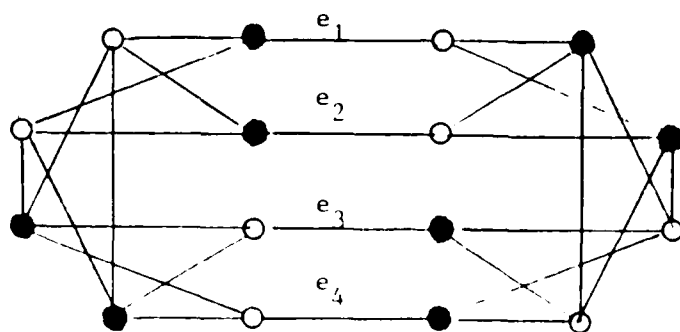
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